

Dual branes in topological sigma models over Lie groups. BF-theory and non-factorizable Lie bialgebras

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ABSTRACT: We continue the study of the Poisson-Sigma model over Poisson-Lie groups. Firstly, we solve the models with targets G and G^* (the dual group of the Poisson-Lie group G) corresponding to a triangular r -matrix and show that the model over G^* is always equivalent to BF-theory. Then, given an arbitrary r -matrix, we address the problem of finding D-branes preserving the duality between the models. We identify a broad class of dual branes which are subgroups of G and G^* , but not necessarily Poisson-Lie subgroups. In particular, they are not coisotropic submanifolds in the general case and what is more, we show that by means of duality transformations one can go from coisotropic to non-coisotropic branes. This fact makes clear that non-coisotropic branes are natural boundary conditions for the Poisson-Sigma model.

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1. Introduction

The Poisson-Sigma model is a two-dimensional topological sigma model defined on a surface Σ and whose target is a Poisson manifold M . The fields of the model are given by a bundle map $(X, \psi) : T\Sigma \rightarrow T^*M$ where $X : \Sigma \rightarrow M$ is the base map and $\psi \in \Gamma(T^*\Sigma \otimes X^*T^*M)$.

The model has given much insight in a variety of topics on Poisson Geometry, since it encodes in a very natural way the geometrical properties of the target. Examples are the relation of the perturbative quantization of the model to Kontsevich's formula for deformation quantization ([15],[7],[9], [4]) as well as the connection to symplectic groupoids ([8]).

A Poisson-Lie group is a Lie group equipped with a Poisson structure which is compatible with the product on the group. The particularization of the model to this case, which we shall call **Poisson-Lie sigma model**, is an interesting topic on its own. Each Poisson-Lie group G has a dual group G^* which is also Poisson-Lie and one expects this duality to show up in the models.

Every Poisson-Lie structure on a complex simple Lie group G is given by an r -matrix either factorizable or triangular (see the following sections). The Poisson-Sigma model in

the factorizable case was studied in [6]. It was discovered a bulk-boundary duality relating the reduced phase space of the models with targets G and G^* when $\Sigma = \mathbb{R} \times [0, \pi]$ and the base map $g : \Sigma \rightarrow M$ is free at the boundary of Σ . Triangular r -matrices were not treated therein due to the absence of an efficient realization of the double group in this case. Now, we can describe explicitly the double group for any Poisson-Lie structure on a complex, simple and simply connected Lie group. We make use of such description for studying in the Lagrangian and Hamiltonian formalisms the Poisson-Sigma model with a triangular Poisson-Lie group as target. We show that the duality found in [6] shows up also in the triangular case.

In [2],[12] it was shown that, in the factorizable case, the Poisson-Sigma model with G^* as target is locally equivalent to the G/G WZW model. We prove in the present paper that when r is triangular, the Poisson-Sigma model with target G^* is always equivalent to BF-theory.

The task of identifying the most general branes (that is, submanifolds to which $X|_{\partial\Sigma}$ can be restricted) which are admissible for the Poisson-Sigma model was started in [9] and completed in [5]. The main objective of the present paper is to apply the machinery of [5] to the Poisson-Lie sigma models over G and G^* and find branes which preserve and generalize the bulk-boundary duality found in [6] in the particular case in which the brane was the whole target group. This duality transformation gave a one-to-one map between the moduli spaces of solutions of the models over G and G^* . The dualizable branes should be compatible with the group and Poisson-Lie structures. We identify a family of dualizable branes which are r -invariant subgroups, but not necessarily coisotropic subgroups. More interestingly, we show that the dual brane of a coisotropic brane can be non-coisotropic. This explains why focusing on pairs of coisotropic branes ([3]) leads to different moduli spaces of solutions.

The paper is organized as follows:

In Section 2 we give a brief survey on Poisson-Lie groups and Lie bialgebras and in Section 3 we describe the double of a Lie bialgebra (and the corresponding double group) adopting an approach that allows to treat both the factorizable and triangular case and understand their differences. Section 4 presents the formulation of the Poisson-Sigma model on surfaces with boundary and a summary of the results of [5] about admissible branes. In Section 5 we recall the results of [6] on the factorizable case and solve in detail the models corresponding to triangular r -matrices. We show that the bulk-boundary duality pointed out in [6] for free boundary conditions is still present in the triangular case. We also prove in the Lagrangian formalism that if r is triangular, the Poisson-Sigma model with target G^* is equivalent to BF-theory. Section 6 deals with the problem of dual branes.

2. Poisson-Lie groups

We present here some basic results on Poisson-Lie groups and Lie bialgebras. See [20],[18] and the introductory notes [16] for details.

A **Poisson manifold** M is a differentiable manifold endowed with a bilinear bracket $\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ verifying:

- (i) $\{f_1, f_2\} = -\{f_2, f_1\}$
 - (ii) $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$ (Jacobi identity)
 - (iii) $\{f_1, f_2 f_3\} = f_2 \{f_1, f_3\} + \{f_1, f_2\} f_3$ (Leibniz rule)
- for any $f_1, f_2, f_3 \in C^\infty(M)$.

The Poisson bracket $\{\cdot, \cdot\}$ determines uniquely a bivector field Π such that:

$$\{f, g\}(p) = \iota(\Pi_p)(df \wedge dg)_p, \quad p \in M$$

Taking local coordinates X^i on M , $\Pi^{ij}(X) = \{X^i, X^j\}$. The Jacobi identity for the Poisson bracket reads in terms of Π^{ij} :

$$\Pi^{ij} \partial_i \Pi^{kl} + \Pi^{ik} \partial_i \Pi^{lj} + \Pi^{il} \partial_i \Pi^{jk} = 0$$

where summation over repeated indices is understood.

For every $p \in M$ define $\Pi_p^\sharp : T_p^* M \rightarrow T_p M$ by

$$\beta(\Pi_p^\sharp(\alpha)) = \iota(\Pi_p)(\alpha \wedge \beta), \quad \alpha, \beta \in T_p^* M$$

Due to the Jacobi identity the image of Π^\sharp ,

$$\text{Im}(\Pi^\sharp) := \bigcup_{p \in M} \text{Im}(\Pi_p^\sharp)$$

is a completely integrable general differential distribution and M admits a generalized foliation ([23]). The Poisson structure restricted to a leaf is non-degenerate and hence defines a symplectic structure on the leaf. That is why one often speaks about the symplectic foliation and the **symplectic leaves** of M .

Let N be a submanifold of the Poisson manifold M with $i : N \hookrightarrow M$ the inclusion map, and denote by $T_p N^0 \subset T_p^* M$ the annihilator of $T_p N, p \in N$. If

$$\Pi_p^\sharp(T_p N^0) \cap T_p N = \{0\}, \quad \forall p \in N \quad (2.1)$$

then for any $p \in N$ there exists a map $\hat{\Pi}_p^\sharp$ that makes the following diagram

$$\begin{array}{ccc} T_p^* N & \xrightarrow{\hat{\Pi}_p^\sharp} & T_p N \\ \uparrow i_p^* & & \downarrow i_{*p} \\ \Pi_p^{\sharp-1}(T_p N) & \xrightarrow{\Pi_p^\sharp} & T_p M \end{array} \quad (2.2)$$

commutative. If the maps $\hat{\Pi}_p^\sharp$ define a smooth bundle map $\hat{\Pi}^\sharp : T^* N \rightarrow TN$ the latter gives a Poisson structure on N (the Dirac bracket) and N is called **Poisson-Dirac submanifold** ([10]). On the other hand, N is said **coisotropic** if $\Pi_p^\sharp(T_p N^0) \subset T_p N, \forall p \in N$. A submanifold N which is both Poisson-Dirac and coisotropic satisfies $\Pi_p^\sharp(T_p N^0) = 0, \forall p \in N$

and is said a **Poisson submanifold**. Equivalently, N is a Poisson submanifold if the inclusion i is a Poisson map.

Consider a Lie group G equipped with a Poisson structure $\{\cdot, \cdot\}_G$. It is natural to demand the Poisson structure to be compatible with the product. G is said to be a **Poisson-Lie group** if the product on the group is a Poisson map, i.e. if

$$\{f, h\}_G(g_1 g_2) = \{f(\cdot g_2), h(\cdot g_2)\}_G(g_1) + \{f(g_1 \cdot), h(g_1 \cdot)\}_G(g_2) \quad (2.3)$$

for $f, h \in C^\infty(G)$. It is evident from (2.3) that a Poisson-Lie structure always vanishes at the unit e of G . Therefore, the linearization of the Poisson structure at e provides a Lie algebra structure on $\mathfrak{g}^* = T_e^*(G)$ by the formula

$$[df_1(e), df_2(e)]_{\mathfrak{g}^*} = d\{f_1, f_2\}_G(e), \quad f_1, f_2 \in C^\infty(G) \quad (2.4)$$

The Poisson-Lie structure of G yields a compatibility condition between the Lie brackets on \mathfrak{g} and \mathfrak{g}^* , namely

$$\begin{aligned} \langle [\xi, \eta]_{\mathfrak{g}^*}, [X, Y] \rangle + \langle \text{ad}_Y^* \eta, \text{ad}_X^* X \rangle - \langle \text{ad}_Y^* \xi, \text{ad}_X^* X \rangle - \\ - \langle \text{ad}_X^* \eta, \text{ad}_Y^* Y \rangle + \langle \text{ad}_X^* \xi, \text{ad}_Y^* Y \rangle = 0. \end{aligned} \quad (2.5)$$

for $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$ and $\langle \cdot, \cdot \rangle$ the natural pairing between elements of a vector space and its dual.¹

The compatibility condition (2.5) between the Lie brackets on \mathfrak{g} and \mathfrak{g}^* defines a **Lie bialgebra** structure for \mathfrak{g} (or, by symmetry, for \mathfrak{g}^*).

Now take $\mathfrak{g} \oplus \mathfrak{g}^*$ with the natural scalar product

$$(X + \xi | Y + \eta) = \langle \eta, X \rangle + \langle \xi, Y \rangle, \quad X, Y \in \mathfrak{g}, \quad \xi, \eta \in \mathfrak{g}^* \quad (2.6)$$

There exists a unique Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ such that \mathfrak{g} and \mathfrak{g}^* are Lie subalgebras and that (2.6) is invariant:

$$[X + \xi, Y + \eta] = [X, Y] + [\xi, \eta]_{\mathfrak{g}^*} - \text{ad}_X^* \eta + \text{ad}_Y^* \xi + \text{ad}_\eta^* X - \text{ad}_\xi^* Y \quad (2.7)$$

The vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ with the Lie bracket (2.7) is called the double of \mathfrak{g} and is denoted by $\mathfrak{g} \bowtie \mathfrak{g}^*$ or \mathfrak{d} .

If G is connected and simply connected, (2.5) is enough to integrate $[\cdot, \cdot]_{\mathfrak{g}^*}$ to a Poisson structure on G that makes it Poisson-Lie and the Poisson structure is unique. Hence, there is a one-to-one correspondence between Poisson-Lie structures on G and Lie bialgebra structures on \mathfrak{g} . The symmetry between \mathfrak{g} and \mathfrak{g}^* in (2.5) implies that one has also a Poisson-Lie group G^* with Lie algebra $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*})$ and a Poisson structure $\{\cdot, \cdot\}_{G^*}$ whose linearization at e gives the Lie bracket of \mathfrak{g} . G^* is the **dual Poisson-Lie group** of G . The connected and simply connected Lie group with Lie algebra $\mathfrak{g} \bowtie \mathfrak{g}^*$ is known as the **double group** of G and denoted by D .

¹ ad^* denotes the coadjoint representation of a Lie algebra on its dual vector space. Hence, $\xi \in \mathfrak{g}^* \mapsto \text{ad}_\xi^*$ is the coadjoint representation of $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*})$ on \mathfrak{g} .

G and G^* are subgroups of D and there exists a neighborhood D_0 of the identity of D such that every element $\nu \in D_0$ can be written as $\nu = ug = \tilde{g}\tilde{u}$, $g, \tilde{g} \in G$, $u, \tilde{u} \in G^*$ and both factorizations are unique (notice that $G_0 := G \cap G^* \subset D$ is a discrete subgroup). These factorizations define a local left action of G^* on G and a local right action of G on G^* by

$$\begin{aligned} {}^u g &= \tilde{g} \\ u^g &= \tilde{u} \end{aligned} \tag{2.8}$$

Starting with the element $gu \in D$ we can define in an analogous way a left action of G on G^* and a right action of G^* on G . These are known as **Dressing transformations** or **Dressing actions** . The symplectic leaves of G (resp. G^*) are the connected components of the orbits of the right or left dressing action of G^* (resp. G).

There is a natural Poisson structure on D which will be important for us since it will show up in the analysis of the reduced phase space of the Poisson-Lie sigma models. Its main symplectic leaf (which contains a neighbourhood of the unit of D) is $D_0 = GG^* \cap G^*G$. We write its inverse in D_0 , which is a symplectic form defined at a point $ug = \tilde{g}\tilde{u} \in D_0$ as:

$$\Omega(ug) = \langle d\tilde{g}\tilde{g}^{-1} \frown duu^{-1} \rangle + \langle g^{-1}dg \frown \tilde{u}^{-1}d\tilde{u} \rangle \tag{2.9}$$

where $\langle \cdot, \cdot \rangle$ acts on the values of the Maurer-Cartan one-forms. D endowed with the Poisson structure yielding Ω is known as **Heisenberg double** ([11],[1]).

A **Poisson-Lie subgroup** $H \subset G$ is a subgroup which is Poisson-Lie and such that the inclusion $i : H \hookrightarrow G$ is a Poisson map. In particular H is a coisotropic submanifold of G . Let us call $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebra of H and $\mathfrak{h}^0 \subset \mathfrak{g}^*$ its annihilator. H is a Poisson-Lie subgroup if and only if \mathfrak{h}^0 is an ideal of \mathfrak{g}^* , i.e. $[\xi, \eta]_{\mathfrak{g}^*} \in \mathfrak{h}^0$, $\forall \xi \in \mathfrak{g}^*, \forall \eta \in \mathfrak{h}^0$. This property permits to restrict the bialgebra structure to \mathfrak{h} , which is then said a **Lie subbialgebra** of \mathfrak{g} . The Poisson-Lie group H associated to \mathfrak{h} is a subgroup of G . However, in general there is no natural way to realize the dual Poisson-Lie group H^* as a subgroup of G^{*2} .

2.1 Poisson-Lie structures on simple Lie groups

Let us take G a complex, simple, connected and simply connected Lie group and give the above construction explicitly. The (essentially unique) nondegenerate, invariant, bilinear form $\text{tr}(\cdot)$ on \mathfrak{g} establishes an isomorphism between \mathfrak{g} and \mathfrak{g}^* . The Poisson structure Π contracted with the right-invariant forms on G , $\theta_R(X) = \text{tr}(dgg^{-1}X)$, $X \in \mathfrak{g}$, will be denoted

$$P_g(X, Y) = \iota(\Pi_g)\theta_R(X) \wedge \theta_R(Y) \tag{2.10}$$

For a general Poisson-Lie structure on G ([18]),

$$P_g^r(X, Y) = \frac{1}{2} \text{tr}(XrY - X\text{Ad}_g r \text{Ad}_g^{-1}Y) \tag{2.11}$$

²In reference [3] H and H^0 (the subgroup corresponding to \mathfrak{h}^0) were proposed as a pair of dual branes for the Poisson-Lie sigma models over G and G^* . We shall see later on in this paper that this is not the right approach. One must take H^* as the dual brane of H .

where $r : \mathfrak{g} \rightarrow \mathfrak{g}$ is an antisymmetric endomorphism such that

$$r[rX, Y] + r[X, rY] - [rX, rY] = \alpha[X, Y], \quad \alpha \in \mathbb{C} \quad (2.12)$$

which is sometimes called **modified Yang-Baxter identity**. Such an operator is what we shall understand by an **r -matrix**³.

It is possible to show that $\text{Ad}_{g_0} r = r \text{Ad}_{g_0}$, $g_0 \in G_0$.

Using r we can define a second Lie bracket on \mathfrak{g} ,

$$[X, Y]_r = \frac{1}{2}([X, rY] + [rX, Y]) \quad (2.13)$$

which is nothing but the linearization of (2.11) at the unit of G . Denoting by \mathfrak{g}_r the vector space \mathfrak{g} equipped with the Lie bracket $[\cdot, \cdot]_r$, we have that \mathfrak{g}_r is isomorphic to $(\mathfrak{g}^*, [\cdot, \cdot]_{\mathfrak{g}^*})$.

In fact, every Lie bialgebra structure on a simple Lie algebra is given by an r -matrix as defined above. The pair (\mathfrak{g}, r) is said a **factorizable** (resp. **non-factorizable** or **triangular**) Lie bialgebra if $\alpha \neq 0$ (resp. $\alpha = 0$). The corresponding Poisson-Lie groups will be said either factorizable or triangular accordingly.

Using the isomorphism given by $\text{tr}(\cdot)$ it is easy to show that $\mathfrak{g} \bowtie \mathfrak{g}^* \cong (\mathfrak{g} \oplus \mathfrak{g}, [\cdot, \cdot]_{\mathfrak{d}})$ as Lie algebras, where

$$\begin{aligned} & [(X, Y), (X', Y')]_{\mathfrak{d}} = \\ & ([X, X'] + \frac{1}{2}([X, rY'] + [rY, X'] + r[Y', X] + r[X', Y]), [X, Y'] + [Y, X'] + [Y, Y']_r) \end{aligned}$$

3. The double of a Lie bialgebra

The study of the Poisson-Sigma model on factorizable Poisson-Lie groups carried out in [6] made use of a concrete realization of the double group. It turns out that the double \mathfrak{d} has a very different aspect for $\alpha \neq 0$ and $\alpha = 0$ and the analysis therein did not cover the triangular case. In this section we rederive an approach (appeared already in [22]) that allows us to understand the cases $\alpha \neq 0$ and $\alpha = 0$ in a unified way.

Consider the Lie algebra $\mathfrak{G} = \mathfrak{g}[[\varepsilon]]$ of polynomials on a variable ε with coefficients in \mathfrak{g} (always a simple complex Lie algebra in this paper) where

$$\left[\sum_{m=0}^M X_m \varepsilon^m, \sum_{n=0}^N X'_n \varepsilon^n \right] = \sum_{m=0}^M \sum_{n=0}^N \varepsilon^{m+n} [X_m, X'_n] \quad (3.1)$$

$\mathfrak{G}_\alpha = (\varepsilon^2 - \alpha)\mathfrak{G}$ is an ideal of \mathfrak{G} , so $\mathfrak{G}/\mathfrak{G}_\alpha$ inherits a Lie algebra structure from \mathfrak{G} . In practice, it is more useful to think of $\mathfrak{G}/\mathfrak{G}_\alpha$ as the set $\{X + Y\varepsilon \mid X, Y \in \mathfrak{g} \mid \varepsilon^2 = \alpha\}$. Then, the Lie bracket of two elements of $\mathfrak{G}/\mathfrak{G}_\alpha$ can be written as

$$[X + Y\varepsilon, X' + Y'\varepsilon] = [X, X'] + \alpha[Y, Y'] + ([X, Y'] + [Y, X'])\varepsilon \quad (3.2)$$

There exists an isomorphism of Lie algebras between $(\mathfrak{g} \oplus \mathfrak{g}, [\cdot, \cdot]_{\mathfrak{d}})$ and $\mathfrak{G}/\mathfrak{G}_\alpha$ given by $(X, Y) \mapsto X + \frac{1}{2}rY + \frac{1}{2}Y\varepsilon$ and, consequently, $\mathfrak{d} \cong \mathfrak{G}/\mathfrak{G}_\alpha$. Furthermore,

$$\mathfrak{g} \cong \{X \mid X \in \mathfrak{g}\}, \quad \mathfrak{g}_r \cong \{rX + X\varepsilon \mid X \in \mathfrak{g}\} \subset \mathfrak{G}/\mathfrak{G}_\alpha$$

³In the literature what we call r is often denoted by R , keeping r for elements of $\mathfrak{g} \otimes \mathfrak{g}$.

3.1 Factorizable Lie bialgebras

It is clear that if $\alpha \neq 0$, the subalgebras $\{(1 + \frac{\varepsilon}{\sqrt{\alpha}})X \mid X \in \mathfrak{g}\}$, $\{(1 - \frac{\varepsilon}{\sqrt{\alpha}})X \mid X \in \mathfrak{g}\}$ commute with one another and both are isomorphic to \mathfrak{g} . In fact, $\mathfrak{G}/\mathfrak{G}_\alpha \cong \mathfrak{g} \oplus \mathfrak{g}$ with the natural Lie bracket:

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2])$$

with the isomorphism given by $X + Y\varepsilon \mapsto (X + Y, X - Y)$. Hence, the double of a factorizable Lie bialgebra is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$ and $D = G \times G$.

As deduced from (2.12) $r_\pm = \frac{1}{2}(r \pm \sqrt{\alpha})$ are Lie algebra morphisms from \mathfrak{g}_r to \mathfrak{g} , i.e.

$$r_\pm[X, Y]_r = [r_\pm X, r_\pm Y]$$

and we have the following embeddings of \mathfrak{g} and \mathfrak{g}_r in $\mathfrak{g} \oplus \mathfrak{g}$:

$$\mathfrak{g}_d = \{(X, X) \mid X \in \mathfrak{g}\}, \quad \mathfrak{g}_r = \{(r_+X, r_-X) \mid X \in \mathfrak{g}\}$$

We shall use the same notation \mathfrak{g}_r for $(\mathfrak{g}, [\cdot, \cdot]_r)$ and for its embedding in $\mathfrak{g} \oplus \mathfrak{g}$. This should not lead to any confusion.

Notice that the map $X \mapsto (r_+X, r_-X)$ is non-degenerate as long as $\alpha \neq 0$. We can recover X by the formula $X = \alpha^{-\frac{1}{2}}(r_+X - r_-X)$. We shall often use the notation $X_\pm = r_\pm X$.

Notice that $\mathfrak{g}_\pm := r_\pm \mathfrak{g}$ are Lie subalgebras of \mathfrak{g} and denote by G_\pm the subgroups of $G \times G$ integrating \mathfrak{g}_\pm . We have the following embeddings of G and G^* :

$$G_d = \{(g, g) \in D \mid g \in G\}, \quad G_r = \{(g_+, g_-) \in D \mid g_+ \in G_+, g_- \in G_-\}$$

The dressing transformations are given by the solutions of the factorization problem:

$$(h_+, h_-)(g, g) = (\tilde{g}, \tilde{g})(\tilde{h}_+, \tilde{h}_-) \quad (3.3)$$

We can write now explicitly the Poisson-Lie structure on G_r dual to (2.11). After contraction with the right-invariant forms on G_r , $\theta_R^r(X) = \text{tr}[(dg_+g_+^{-1} - dg_-g_-^{-1})X]$ for $X \in \mathfrak{g}$, it takes the form

$$P_{(g_+, g_-)}^r(X, Y) = \text{tr} \left(X(\text{Ad}_{g_+} - \text{Ad}_{g_-})(r_- \text{Ad}_{g_+}^{-1} - r_+ \text{Ad}_{g_-}^{-1})Y \right) \quad (3.4)$$

which verifies, in particular, that its linearization at the unit of G_r gives the Lie bracket of \mathfrak{g} .

Using the explicit description of the double group we can write the symplectic structure (2.9) at a point $(h_+g, h_-g) = (\tilde{g}\tilde{h}_+, \tilde{g}\tilde{h}_-) \in D_0$ as:

$$\Omega((h_+g, h_-g)) = \text{tr} \left(d\tilde{g}\tilde{g}^{-1} \wedge (dh_+h_+^{-1} - dh_-h_-^{-1}) + g^{-1}dg \wedge (\tilde{h}_+^{-1}d\tilde{h}_+ - \tilde{h}_-^{-1}d\tilde{h}_-) \right)$$

Example:

Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} and ϕ its set of roots. The decomposition in root spaces reads

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \phi} \mathfrak{g}_\alpha \right) \quad (3.5)$$

where \mathfrak{t} is a Cartan subalgebra of \mathfrak{g} and

$$\mathfrak{g}_\alpha = \{ \mathbb{C}X_\alpha \mid [T, X_\alpha] = \alpha(T)X_\alpha, \forall T \in \mathfrak{t} \}$$

Given a splitting into positive and negative roots, $\phi = \phi_+ \cup \phi_-$, any element $X \in \mathfrak{g}$ can be written as $X = X^{(+)} + X^{(-)} + T$ where $X^{(\pm)} \in \text{Span}(X_\alpha)$, $\alpha \in \phi_\pm$.

The **standard r -matrix** is defined by $r = r_+ + r_-$ with

$$r_+ X = X^{(+)} + \frac{1}{2}T \quad (3.6)$$

$$r_- X = -X^{(-)} - \frac{1}{2}T \quad (3.7)$$

which is a factorizable r -matrix with $\alpha = 1$.

Take as a particular case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ with the standard r -matrix. Then, $\mathfrak{sl}(n, \mathbb{C})_r \subset \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$ consists of pairs (X_+, X_-) where X_+ (resp. X_-) is upper (resp. lower) triangular and $\text{diag}(X_+) = -\text{diag}(X_-)$.

At the group level, $SL(n, \mathbb{C})_r \subset SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$ is the set of pairs (g_+, g_-) such that g_+ (resp. g_-) is upper (resp. lower) triangular and $\text{diag}(g_+) = \text{diag}(g_-)^{-1}$.

3.2 Triangular Lie bialgebras

In this subsection we describe the double of a triangular Lie bialgebra and the double of the associated Poisson-Lie groups. We shall use these results for writing explicitly the Poisson structure dual to (2.11) and, in the subsequent sections, for solving the corresponding Poisson-Lie sigma models.

If $\alpha = 0$, r_\pm degenerate to $\frac{1}{2}r$, the map $X \mapsto (\frac{1}{2}rX, \frac{1}{2}rX)$ is not invertible and $\mathfrak{G}/\mathfrak{G}_0$ is no longer isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$. Indeed, $\mathfrak{G}/\mathfrak{G}_0 = \{X + Y\varepsilon \mid \varepsilon^2 = 0\}$ is not semisimple, for the elements $X\varepsilon$ form an abelian ideal as seen from the Lie bracket

$$[X + Y\varepsilon, X' + Y'\varepsilon] = [X, X'] + ([X, Y'] + [Y, X'])\varepsilon \quad (3.8)$$

This is the Lie algebra of the tangent bundle of G , $TG \cong G \times \mathfrak{g}$, with the natural group structure given by the semidirect product:

$$(g, X)(g', X') = (gg', \text{Ad}_g X' + X) \quad (3.9)$$

Hence, the double of a triangular Lie bialgebra is isomorphic to the tangent bundle of G with the product given by (3.9).

We can represent the elements of the double as

$$D = \left\{ \begin{pmatrix} e & 0 \\ X & e \end{pmatrix} g \mid g \in G, X \in \mathfrak{g} \right\} \quad (3.10)$$

where the product is now the formal product of matrices, resulting the semidirect product mentioned above. Its Lie algebra with this notation is

$$\mathfrak{d} = \left\{ \begin{pmatrix} X & 0 \\ Y & X \end{pmatrix} \mid X, Y \in \mathfrak{g} \right\} \quad (3.11)$$

and the Lie bracket (3.8) is given by the formal commutator of matrices. The embeddings of \mathfrak{g} and \mathfrak{g}_r in \mathfrak{d} are given by:

$$\mathfrak{g}_d = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \mid X \in \mathfrak{g} \right\}, \quad \mathfrak{g}_r = \left\{ \begin{pmatrix} rX & 0 \\ X & rX \end{pmatrix} \mid X \in \mathfrak{g} \right\} \quad (3.12)$$

Both subalgebras exponentiate to subgroups $G_d, G_r \subset D$. Clearly,

$$G_d = \left\{ \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mid g \in G \right\} \quad (3.13)$$

whereas for G_r the description is less explicit. It is the subgroup of D generated by elements of the form:

$$\begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} e^{rX} \quad \text{with} \quad Y = \int_0^1 \text{Ad}_{e^{srX}} X ds, \quad X \in \mathfrak{g}. \quad (3.14)$$

We will denote a general element of G_r by

$$\bar{Y} = \begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} h_Y \quad (3.15)$$

where, in the general case, Y belongs to a dense subset of \mathfrak{g} and determines h_Y up to multiplication by an element of G_0 . This means that

$$\text{if } \begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} h_Y \in G_r, \text{ then}$$

$$\begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} h'_Y \in G_r \Leftrightarrow h'_Y = h_Y g_0, \quad g_0 \in G_0.$$

As a consequence, the notation \bar{Y} has a small ambiguity, but we shall use of it for brevity wherever it does not lead to confusion.

The realization of G^* described above allows us to write the Poisson structure dual to (2.11).

The right-invariant forms in G_r are

$$\theta_R^r(Y) = \text{tr} \left((dX + [X, dh_X h_X^{-1}]) Y \right) = \text{tr} \left(d \left(\text{Ad}_{h_X}^{-1} X \right) \text{Ad}_{h_X}^{-1} Y \right)$$

for $Y \in \mathfrak{g}$ and $\begin{pmatrix} e & 0 \\ X & e \end{pmatrix} h_X \in G_r$. Whereas the left invariant forms read:

$$\theta_L^r(Y) = \text{tr} \left(Y \text{Ad}_{h_X}^{-1} dX \right).$$

It can be checked after a straightforward (although lengthy) calculation that the dual Poisson-Lie structure contracted with the right-invariant forms is

$$P_X^r(Y, Z) = \text{tr} \left(Y[X, Z] - [X, Y] \text{Ad}_{h_X} r \text{Ad}_{h_X}^{-1} [X, Z] \right) \quad (3.16)$$

The symplectic structure on the Heisenberg double (2.9) at a point $\bar{X}g = \tilde{g}\tilde{X} \in D_0$ can be written now:

$$\Omega(\bar{X}g) = \text{tr} \left(d\tilde{g}\tilde{g}^{-1} \wedge (dX + [X, dh_X h_X^{-1}]) + g^{-1}dg \wedge \text{Ad}_{h_X}^{-1} d\tilde{X} \right) \quad (3.17)$$

Example:

Take \mathfrak{g} a complex simple Lie algebra. If $\tau_{\mathfrak{t}} : \mathfrak{g} \rightarrow \mathfrak{t}$ is the projector onto the Cartan subalgebra \mathfrak{t} with respect to the decomposition (3.5) and $\mathcal{O} : \mathfrak{t} \rightarrow \mathfrak{t}$ is an antisymmetric endomorphism of \mathfrak{t} with respect to $\text{tr}(\cdot)$, then $r = \mathcal{O}\tau_{\mathfrak{t}}$ is an r -matrix with $\alpha = 0$.

It is worth studying in detail the structure of G_r for this r . As we know, G_r is generated by elements of the form

$$\begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} e^{rX}, \quad Y = \int_0^1 \text{Ad}_{e^{srX}} X ds, \quad X \in \mathfrak{g} \quad (3.18)$$

The elements $\{rX \mid X \in \mathfrak{g}\}$ span a subalgebra of \mathfrak{t} (therefore abelian) and its exponentiation will be a subgroup of the Cartan subgroup of G , so we concentrate on Y .

Take

$$X = T + \sum_{\alpha \in \phi} a_{\alpha} X_{\alpha}, \quad T \in \mathfrak{t}$$

Using that

$$\text{Ad}_{e^{srX}} = e^{ad(srX)}$$

we straightforwardly obtain

$$\text{Ad}_{e^{srX}} X = T + \sum_{\alpha \in \phi} e^{s\alpha(rT)} a_{\alpha} X_{\alpha} \quad (3.19)$$

and, therefore,

$$Y = T + \sum_{\alpha \in \phi} \frac{1}{\alpha(rT)} \left(e^{\alpha(rT)} - 1 \right) a_{\alpha} X_{\alpha} \quad (3.20)$$

where, if some $\alpha(rT) = 0$, the limit $\alpha(rT) \rightarrow 0$ must be understood in the last expression.

The product of n elements of the form (3.18)

$$\begin{pmatrix} e & 0 \\ Y & e \end{pmatrix} T_Y = \begin{pmatrix} e & 0 \\ Y_1 & e \end{pmatrix} e^{rX_1} \cdots \begin{pmatrix} e & 0 \\ Y_n & e \end{pmatrix} e^{rX_n}$$

yields

$$T_Y = e^{rT_1 + \cdots + rT_n}$$

$$\begin{aligned}
Y = & T_1 + \cdots + T_n + \sum_{\alpha \in \phi} \left[\frac{1}{\alpha(rT_1)} \left(e^{\alpha(rT_1 + \cdots + rT_n)} - e^{\alpha(rT_2 + \cdots + rT_n)} \right) a_{\alpha,1} + \right. \\
& + \frac{1}{\alpha(rT_2)} \left(e^{\alpha(rT_2 + \cdots + rT_n)} - e^{\alpha(rT_3 + \cdots + rT_n)} \right) a_{\alpha,2} + \\
& \left. + \cdots + \frac{1}{\alpha(rT_n)} \left(e^{\alpha(rT_n)} - 1 \right) a_{\alpha,n} \right] X_\alpha
\end{aligned} \tag{3.21}$$

where

$$X_i = T_i + \sum_{\alpha \in \phi} a_{\alpha,i} X_\alpha, \quad T_i \in \mathfrak{t}$$

From (3.21) it is clear that, in this case, $G_0 = e$ and that Y fills \mathfrak{g} . As a consequence, the dressing actions are globally defined, $D = GG_r = G_r G$ and the factorizations are unique.

Example:

$r = 0$ is an r -matrix with $\alpha = 0$. It endows G with trivial Poisson bracket $\{\cdot, \cdot\}_G = 0$ and \mathfrak{g}^* with trivial Lie bracket $[\cdot, \cdot]_{\mathfrak{g}^*} = 0$. The dual Poisson Lie group G^* is \mathfrak{g}^* viewed as an abelian group and equipped with the so-called **Kostant-Kirillov Poisson bracket**, namely:

$$\{X, Y\}_{\mathfrak{g}^*}(\xi) = \langle \xi, [X, Y] \rangle, \quad X, Y \in \mathfrak{g}, \quad \xi \in \mathfrak{g}^* \tag{3.22}$$

for linear functions on \mathfrak{g}^* (elements of \mathfrak{g}) and extended by the Leibniz rule to $C^\infty(\mathfrak{g}^*)$.

4. The Poisson-Sigma model

The Poisson-Sigma model ([19]) is a two-dimensional topological sigma model with a Poisson manifold (M, Π) as target. The fields of the model are $X : \Sigma \rightarrow M$ and a 1-form ψ on Σ with values in the pull-back by X of the cotangent bundle of M . The action functional has the form

$$S_{P\sigma}(X, \psi) = \int_{\Sigma} \langle dX, \wedge \psi \rangle - \frac{1}{2} \langle \Pi \circ X, \psi \wedge \psi \rangle \tag{4.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between vectors and covectors of M .

If X^i are local coordinates on M , σ^μ local coordinates on Σ , $\Pi^{ij}(X)$ the components of the Poisson structure in these coordinates and $\psi_i = \psi_{i\mu} d\sigma^\mu$, $i = 1, \dots, n; \mu = 1, 2$ the action reads

$$S_{P\sigma}(X, \psi) = \int_{\Sigma} dX^i \wedge \psi_i - \frac{1}{2} \Pi^{ij}(X) \psi_i \wedge \psi_j \tag{4.2}$$

The equations of motion in the bulk are:

$$dX^i + \Pi^{ij}(X) \psi_j = 0 \tag{4.3a}$$

$$d\psi_i + \frac{1}{2} \partial_i \Pi^{jk}(X) \psi_j \wedge \psi_k = 0 \tag{4.3b}$$

In particular, X lies within one of the symplectic leaves of the foliation of M .
Take $\epsilon = \epsilon_i dX^i$ a section of X^*T^*M . The infinitesimal transformation

$$\begin{aligned}\delta_\epsilon X^i &= \epsilon_j \Pi^{ji}(X) \\ \delta_\epsilon \psi_i &= d\epsilon_i + \partial_i \Pi^{jk}(X) \psi_j \epsilon_k\end{aligned}\tag{4.4}$$

leaves the action (4.2) invariant up to a boundary term

$$\delta_\epsilon S_{P\sigma} = - \int_\Sigma d(dX^i \epsilon_i).\tag{4.5}$$

Notice that the gauge transformations (4.4) form an open algebra, as the commutator of two of them closes only on-shell:

$$[\delta_\epsilon, \delta_{\epsilon'}]X^i = \delta_{[\epsilon, \epsilon']^*}X^i\tag{4.6a}$$

$$[\delta_\epsilon, \delta_{\epsilon'}]\psi_i = \delta_{[\epsilon, \epsilon']^*}\psi_i + \epsilon_k \epsilon'_l \partial_i \partial_j \Pi^{kl}(dX^j + \Pi^{js}(X)\psi_s)\tag{4.6b}$$

where $[\epsilon, \epsilon']_k^* = \partial_k \Pi^{ij}(X) \epsilon_i \wedge \epsilon'_j$.

The general study of the model defined on a surface with boundary was carried out in [5] and [4]. Assume that X at the boundary of Σ is restricted to a submanifold (brane) $N \hookrightarrow M$. Define $\mathcal{I}_N := \{f \in C^\infty(M) | f(p) = 0, p \in N\}$ and $\mathcal{F}_N := \{f \in C^\infty(M) | \{f, \mathcal{I}_N\} \subset \mathcal{I}_N\}$. The brane N is **classically admissible**⁴ if it satisfies the regularity condition

$$\dim\{(df)_p | f \in \mathcal{F}_N \cap \mathcal{I}_N\} = \text{const.}, \forall p \in N\tag{4.7}$$

If N is classically admissible the Poisson-Sigma model with the boundary condition $X|_{\partial\Sigma} \in N$ is consistent. The appropriate boundary condition for ψ is that its contraction with vectors tangent to $\partial\Sigma$, ψ_t , takes values in $\{df | f \in \mathcal{F}_N \cap \mathcal{I}_N\}$, and the gauge transformations are restricted at the boundary to the same space.

5. Poisson-Lie sigma models

In this section we recall the results of ([6]) for factorizable Poisson-Lie groups and treat in detail the triangular case.

When the target manifold is a Lie group, the action of the Poisson-Sigma model can be recasted in terms of a set of fields adapted to the group structure. T^*G can be identified, by right translations, with $G \times \mathfrak{g}^*$ and, using $\text{tr}(\cdot)$, with $G \times \mathfrak{g}$. Then, in (4.1) we can take $A \in \Lambda^1(\Sigma) \otimes \mathfrak{g}$ (instead of ψ) and $g : \Sigma \rightarrow G$ as fields and use the Poisson structure contracted with the right-invariant forms in G (2.10). Denoting by $P_g^\sharp : \mathfrak{g} \rightarrow \mathfrak{g}$ the endomorphism induced by P_g using $\text{tr}(\cdot)$ we can write the action of the Poisson-Sigma model as

$$S(g, A) = \int_\Sigma \text{tr}(dg g^{-1} \wedge A) - \frac{1}{2} \text{tr}(A \wedge P_g^\sharp A)\tag{5.1}$$

⁴The quantization of the model on the disk with a general brane was discussed in [4]. Therein, a regularity condition stronger than (4.7) was imposed to the brane.

In particular, for the Poisson-Lie structure (2.11) we have

$$S_{PL}(g, A) = \int_{\Sigma} \text{tr}(\text{d}g g^{-1} \wedge A) - \frac{1}{4} \text{tr}(A \wedge (r - \text{Ad}_g r \text{Ad}_g^{-1})A) \quad (5.2)$$

which is the action of what we shall call **Poisson-Lie sigma model** with target G .

The equations of motion are

$$\text{d}g g^{-1} + \frac{1}{2}(r - \text{Ad}_g r \text{Ad}_g^{-1})A = 0 \quad (5.3a)$$

$$\text{d}\tilde{A} + [\tilde{A}, \tilde{A}]_r = 0, \quad \tilde{A} := \text{Ad}_g^{-1}A \quad (5.3b)$$

from which a zero curvature equation can be also derived for A ,

$$\text{d}A + [A, A]_r = 0, \quad (5.4)$$

The infinitesimal gauge symmetry of the action, for $\beta : \Sigma \rightarrow \mathfrak{g}$ is

$$\delta_{\beta} g g^{-1} = \frac{1}{2}(\text{Ad}_g r \text{Ad}_g^{-1} - r)\beta \quad (5.5a)$$

$$\delta_{\beta} A = \text{d}\beta + [A, \beta]_r - \frac{1}{2}[\text{d}g g^{-1} + \frac{1}{2}(r - \text{Ad}_g r \text{Ad}_g^{-1})A, \beta] \quad (5.5b)$$

which corresponds to the right dressing vector fields of [18] translated to the origin by right multiplication in G . Its integration (local as in general the vector field is not complete) gives rise to the dressing transformation of g . On-shell, $[\delta_{\beta_1}, \delta_{\beta_2}] = \delta_{[\beta_1, \beta_2]_r}$ and we can talk properly about a gauge group. In fact, the gauge group of the Poisson-Sigma model on G is its dual G^* .

Up to here we have not needed to distinguish between $\alpha = 0$ and $\alpha \neq 0$. In order to study further (5.2) and its dual model we need to make use of the embeddings of G and G^* in the double D . As we have learnt, D is very different in the factorizable and triangular cases and we must analyse them separately.

5.1 Factorizable Poisson-Lie sigma models

It was shown in ([6]) that locally the solutions in the bulk are:

$$A = h_+ \text{d}h_+^{-1} - h_- \text{d}h_-^{-1} \quad (5.6)$$

and $g(\sigma)$ is given by the solution of

$$(h_+(\sigma)\hat{g}, h_-(\sigma)\hat{g}) = (g(\sigma)\tilde{h}_+(\sigma), g(\sigma)\tilde{h}_-(\sigma)) \quad (5.7)$$

We go on to study the reduced phase space of the model when $\Sigma = \mathbb{R} \times [0, \pi]$ and g is free at the boundary. Equivalently, in the language of Section 4, we take a brane which is the whole target manifold $N = G$. The A field must then vanish on vectors tangent to $\partial\Sigma$, so that h_{\pm}, \tilde{h}_{\pm} are constant along the connected components of the boundary. Writing $\sigma = (t, x)$, we have $h_{\pm}(t, 0) = h_{0\pm}, \tilde{h}_{\pm}(t, 0) = \tilde{h}_{0\pm}, h_{\pm}(t, \pi) = h_{\pi\pm}, \tilde{h}_{\pm}(t, \pi) = \tilde{h}_{\pi\pm}$.

Denote $I = [0, \pi]$. The canonical symplectic form on the space of continuous maps $(g, A_x) : TI \rightarrow G \times \mathfrak{g}$ with continuously differentiable base map is:

$$\omega = \int_0^\pi \text{tr}(\delta g g^{-1} \wedge \delta g g^{-1} A_x - \delta g g^{-1} \wedge \delta A_x) dx$$

When restricted to the solutions of the equations of motion the symplectic form ω becomes degenerate, its kernel given by the gauge transformations (5.5) which vanish at $x = 0, \pi$. By definition, the reduced phase space⁵ $\mathcal{P}(G, G)$ is the (possibly singular) quotient of the space of solutions by the kernel of ω .

If we parametrize the solutions in terms of $h_\pm(\sigma), \hat{g}$ we obtain

$$\omega = \frac{1}{2} \int_0^\pi \partial_x \Omega((h_+(\sigma)\hat{g}, h_-(\sigma)\hat{g})) dx \quad (5.8)$$

That is, ω depends only on the values of the fields at the boundary (i.e. the degrees of freedom of the theory are all at the boundary, as expected from the topological nature of the model) and is expressed in terms of the symplectic structure on the Heisenberg double (3.5). Namely,

$$\omega = \frac{1}{2} [\Omega((h_{\pi+}\hat{g}, h_{\pi-}\hat{g})) - \Omega((h_{0+}\hat{g}, h_{0-}\hat{g}))]$$

Or if we take $\sigma_0 = (t_0, 0)$, i.e. $h_{0\pm} = \tilde{h}_{0\pm} = e$

$$\omega = \frac{1}{2} \Omega((h_{\pi+}\hat{g}, h_{\pi-}\hat{g}))$$

The reduced phase space $\mathcal{P}(G, G)$ is then the set of pairs $([(h_+, h_-)], \hat{g})$ with $[(h_+, h_-)]$ a homotopy class of maps from $[0, \pi]$ to G_r which are the identity at 0 and have fixed value at π and such that $(h_+(x)\hat{g}, h_-(x)\hat{g}) \in D_0$, $x \in [0, \pi]$. The symplectic form on $\mathcal{P}(G, G)$ can be viewed as the pull-back of Ω by the map $[(h_+, h_-), \hat{g}] \mapsto (h_{\pi+}\hat{g}, h_{\pi-}\hat{g})$.

5.1.1 The dual factorizable model

Using the Poisson structure given in (3.4) the action of the dual model reads

$$\begin{aligned} S_{PL}^*(g_+, g_-, A) = & \int_{\Sigma} \text{tr}[(dg_+ g_+^{-1} - dg_- g_-^{-1}) \wedge A + \\ & + \frac{1}{2} A \wedge (\text{Ad}_{g_+} - \text{Ad}_{g_-})(r_+ \text{Ad}_{g_-}^{-1} - r_- \text{Ad}_{g_+}^{-1}) A] \end{aligned} \quad (5.9)$$

As shown in ([12],[2]) the Poisson-Lie sigma model with target G_r and fields (g_+, g_-) and A is locally equivalent to the G/G WZW model with fields $g = g_- g_+^{-1}$ and A . This relation can be established for any factorizable r -matrix.

The equations of motion of the model can be written

$$\begin{aligned} g_\pm^{-1} dg_\pm + r_\pm (\text{Ad}_{g_+}^{-1} - \text{Ad}_{g_-}^{-1}) A &= 0 \\ dA + [A, A] &= 0 \end{aligned} \quad (5.10)$$

⁵ $\mathcal{P}(M, N)$ stands for the reduced phase space of the Poisson-Sigma model with target M and brane N .

The gauge transformations, for $\beta : \Sigma \rightarrow \mathfrak{g}$, read:

$$g_{\pm}^{-1} \delta_{\beta} g_{\pm} = r_{\pm} (\text{Ad}_{g_{-}}^{-1} - \text{Ad}_{g_{+}}^{-1}) \beta \quad (5.11a)$$

$$\begin{aligned} \delta_{\beta} A = & d\beta + [A, \beta] + \\ & + \frac{1}{2} (r_{+} \text{Ad}_{g_{-}} + r_{-} \text{Ad}_{g_{+}}) [g_{+}^{-1} dg_{+} - g_{-}^{-1} dg_{-} + (\text{Ad}_{g_{+}}^{-1} - \text{Ad}_{g_{-}}^{-1}) A, \tilde{\beta}] \end{aligned} \quad (5.11b)$$

where $\tilde{\beta} := (r_{-} \text{Ad}_{g_{+}}^{-1} - r_{+} \text{Ad}_{g_{-}}^{-1}) \beta$. The gauge transformations close on shell. Namely, $[\delta_{\beta}, \delta_{\gamma}] = \delta_{[\beta, \gamma]}$ which corresponds now to the gauge group G .

The solutions of the equations of motion can be obtained along the same lines as before. Locally,

$$A = h dh^{-1} \quad (5.12)$$

And $(g_{+}(\sigma), g_{-}(\sigma))$ is obtained as the solution of:

$$(g_{+}(\sigma) \tilde{h}(\sigma), g_{-}(\sigma) \tilde{h}(\sigma)) = (h(\sigma) \hat{g}_{+}, h(\sigma) \hat{g}_{-}),$$

which means that (g_{+}, g_{-}) is the dressing-transformed of $(\hat{g}_{+}, \hat{g}_{-})$ by h . At this point it is evident the symmetry between both dual models under the exchange of the roles of G and G_r .

In the open geometry with free boundary conditions h is constant along connected components of the boundary and one may take $h(t, 0) = \tilde{h}(t, 0) = e$, $h(t, \pi) = h_{\pi}$, $\tilde{h}(t, \pi) = \tilde{h}_{\pi}$. The symplectic form can then be written

$$\omega^{*} = \frac{1}{2} \Omega((h_{\pi} \hat{g}_{+}, h_{\pi} \hat{g}_{-})). \quad (5.13)$$

The duality between $\mathcal{P}(G, G)$ and $\mathcal{P}(G_r, G_r)$ was pointed out in [6]. The symplectic forms of the two models coincide upon the exchange of h_{π} with \hat{g}^{-1} and $(\hat{g}_{+}, \hat{g}_{-})$ with $(h_{\pi+}^{-1}, h_{\pi-}^{-1})$. Hence, one can talk about a bulk-boundary duality between the Poisson-Lie sigma models for G and G^{*} since the exchange of degrees of freedom maps variables associated to the bulk of one model to variables associated to the boundary of the other one.

In the next subsection we make use of the explicit realization of the double in the triangular case given in Section 3.2 for solving the corresponding Poisson-Lie sigma models. We shall see that the bulk-boundary duality found in the factorizable case still holds.

5.2 Triangular Poisson-Lie sigma models

We now go back to equations (5.10) and assume r is triangular ($\alpha = 0$). We start noting that whereas A is a pure gauge of the group G_r , $\frac{1}{2}rA$ is a pure gauge of the group G , i.e.

$$d(\frac{1}{2}rA) + [\frac{1}{2}rA, \frac{1}{2}rA] = 0$$

Now take

$$\bar{X}(\sigma) = \begin{pmatrix} e & 0 \\ X(\sigma) & e \end{pmatrix} h_X(\sigma) \in G_r$$

Then, locally,

$$\bar{X}^{-1} d\bar{X} = \begin{pmatrix} h_X^{-1} dh_X & 0 \\ h_X^{-1} dX h_X & h_X^{-1} dh_X \end{pmatrix} = \begin{pmatrix} rA & 0 \\ A & rA \end{pmatrix} \quad (5.14)$$

\tilde{A} is also a pure gauge of the group G_r , so $\tilde{A} = h_{\tilde{X}}^{-1} d\tilde{X} h_{\tilde{X}}$ and the equation of motion for g reads,

$$dg g^{-1} + \frac{1}{2} (h_X^{-1} dh_X - \text{Ad}_g h_{\tilde{X}}^{-1} dh_{\tilde{X}}) = 0 \quad (5.15)$$

which implies

$$g = h_X^{-1} \hat{g} h_{\tilde{X}}, \quad \hat{g} \in G \quad (5.16)$$

$\tilde{A} = \text{Ad}_g^{-1} A \Rightarrow \tilde{X} = \text{Ad}_{\hat{g}}^{-1} X$ and we can write, locally, the solution as an equation in the double:

$$\begin{pmatrix} e & 0 \\ X & e \end{pmatrix} h_X \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} g = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \hat{g} \begin{pmatrix} e & 0 \\ \tilde{X} & e \end{pmatrix} h_{\tilde{X}} \quad (5.17)$$

Therefore, in the language of dressing actions, $g = \bar{X}^{-1} \hat{g}$.

The analysis of the reduced phase space when $\Sigma = \mathbb{R} \times [0, \pi]$ and $g|_{\partial\Sigma}$ is free works as in the factorizable case. The field A must vanish on vectors tangent to $\partial\Sigma$ and hence \bar{X} is constant along each connected component of the boundary.

By using the explicit solution (5.17) we can identify $\mathcal{P}(G, G)$. Notice that we can always choose $\bar{X}(t, 0) = \tilde{X}(t, 0) = e$. With this choice and defining $X_\pi := X(t, \pi)$, $\tilde{X}_\pi := \tilde{X}(t, \pi)$, $g_\pi := g(t, \pi)$, a straightforward calculation yields

$$\omega = \frac{1}{2} \text{tr} \left(\delta X_\pi + [X_\pi, \delta h_{X_\pi} h_{X_\pi}^{-1}] \right) \wedge \delta \hat{g} \hat{g}^{-1} + \text{Ad}_{h_{\tilde{X}_\pi}^{-1}} \delta \tilde{X}_\pi \wedge g_\pi^{-1} \delta g_\pi = \quad (5.18)$$

$$= \frac{1}{2} \Omega(\bar{X}_\pi \hat{g}) \quad (5.19)$$

The reduced phase space $\mathcal{P}(G, G)$ turns out to be the set of pairs $([\bar{X}], \hat{g})$ with $[\bar{X}]$ a homotopy class of maps from $[0, \pi]$ to G_r which are the identity at $x = 0$ and have fixed value at $x = \pi$.

5.2.1 The dual triangular model and BF-theory

Take (3.16) and write the action of the Poisson-Sigma model with target G_r :

$$\begin{aligned} S_{PL}^*(\bar{X}, A) &= \int_\Sigma \text{tr} \left((dX \wedge A + dX \wedge \text{Ad}_{g_X} r \text{Ad}_{g_X}^{-1} [X, A] + \frac{1}{2} A \wedge [X, A] - \right. \\ &\quad \left. - \frac{1}{2} [X, A] \wedge \text{Ad}_{g_X} r \text{Ad}_{g_X}^{-1} [X, A]) \right) \end{aligned} \quad (5.20)$$

with fields $A \in \Lambda^1(\Sigma) \otimes \mathfrak{g}$, $\bar{X} : \Sigma \rightarrow G_r$.

Note that the action is actually determined by X and A , since it is invariant under $g_X \mapsto g_X g_0$, $g_0 \in G_0$.

Varying the action with respect to A we get the equation of motion for X ,

$$dX + [A, X] = 0 \quad (5.21)$$

Taking variations with respect to \bar{X} and after a rather cumbersome calculation we obtain the equation of motion for A ,

$$dA + [A, A] = 0 \quad (5.22)$$

The infinitesimal gauge symmetry for $\beta : \Sigma \rightarrow \mathfrak{g}$

$$\delta_\beta X = [X, \beta] \quad (5.23a)$$

$$\delta_\beta A = d\beta + [A, \beta] - r[dX + [A, X], \beta + \text{Ad}_{g_X} r \text{Ad}_{g_X}^{-1} [X, \beta]] \quad (5.23b)$$

which this time corresponds to the vector fields of the infinitesimal form of the right dressing action of G on G^* . On-shell, $[\delta_{\beta_1}, \delta_{\beta_2}] = \delta_{[\beta_1, \beta_2]}$

The solutions of the equations of motion are, locally,

$$\begin{aligned} A &= h^{-1} dh \\ X &= \text{Ad}_h^{-1} \hat{X} \end{aligned} \quad (5.24)$$

with $h : U \rightarrow G$, $U \subset \Sigma$ an open contractible subset and $\hat{X} \in \mathfrak{g}$.

This can also be written as an equation in D :

$$\begin{pmatrix} e & 0 \\ \hat{X} & e \end{pmatrix} g_{\hat{X}} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \tilde{h} = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} h \begin{pmatrix} e & 0 \\ X & e \end{pmatrix} g_X \quad (5.25)$$

which can be obtained from (5.17) taking $X \rightarrow \hat{X}$, $\hat{g} \rightarrow h$, $g \rightarrow \tilde{h}$, $\tilde{Y} \rightarrow Y$. Now, $\bar{X} = h^{-1} \hat{X}$.

Now consider $\Sigma = \mathbb{R} \times [0, \pi]$ and $\bar{X}|_{\partial\Sigma}$ free. h must be constant along each connected component of the boundary.

By choosing $h(t, 0) = \tilde{h}(t, 0) = e$, defining $h_\pi := h(t, \pi)$, $\tilde{h}_\pi := \tilde{h}(t, \pi)$, $X_\pi := X(t, \pi)$ and plugging in (5.24), we get

$$\omega^* = \frac{1}{2} \text{tr} \left((\delta \hat{X} + [\hat{X}, \delta \hat{g}_X \hat{g}_X^{-1}]) \wedge \delta \tilde{h}_\pi \tilde{h}_\pi^{-1} + \text{Ad}_{g_{X_\pi}}^{-1} \delta X_\pi \wedge \tilde{h}_\pi^{-1} \delta \tilde{h}_\pi \right) = \quad (5.26)$$

$$= \frac{1}{2} \Omega(h_\pi \hat{X}) \quad (5.27)$$

The reduced phase space $\mathcal{P}(G_r, G_r)$ is the set of pairs $([h], \hat{X})$ with $[h]$ a homotopy class of maps from $[0, \pi]$ to G which are the identity at $x = 0$ and have fixed value at $x = \pi$. Notice the duality between $\mathcal{P}(G, G)$ and $\mathcal{P}(G_r, G_r)$ under the interchange $\hat{g} \leftrightarrow h_\pi$, $X_\pi \leftrightarrow \hat{X}$. This is the triangular version of the bulk-boundary duality found in [6] and recalled in Section 5.1 for the factorizable case.

Now, consider as target of the Poisson-Sigma model the dual of a simple complex Lie algebra \mathfrak{g}^* with the Kostant-Kirillov Poisson bracket. As mentioned in Section 3.2 this is the dual Poisson-Lie group of the simply connected Lie group G whose Lie algebra is \mathfrak{g} endowed with the zero Poisson structure. The action in this particular case is:

$$S_{\text{BF}} = \int_{\Sigma} \text{tr} \left(dX \wedge A - \frac{1}{2} [X, A] \wedge A \right) \quad (5.28)$$

with $X : \Sigma \rightarrow \mathfrak{g}$ and $A \in \Lambda^1(\Sigma) \otimes \mathfrak{g}$. This is the action of BF-theory ([13]) up to a boundary term.

The equations of motion are

$$\begin{aligned} dX + [A, X] &= 0 \\ dA + [A, A] &= 0 \end{aligned} \quad (5.29)$$

For $\epsilon \in \Lambda^0(M) \otimes \mathfrak{g}$ the gauge transformation

$$\delta_{\epsilon} X = [X, \epsilon] \quad (5.30a)$$

$$\delta_{\epsilon} A = d\epsilon + [A, \epsilon] \quad (5.30b)$$

induces the change of the action (5.28) by a boundary term

$$\delta_{\epsilon} S_{\text{BF}} = - \int_{\Sigma} \text{dtr}(dX \epsilon). \quad (5.31)$$

Note that in this case the gauge transformations close even off-shell

$$[\delta_{\epsilon}, \delta_{\epsilon'}] = \delta_{[\epsilon, \epsilon']}$$

and induce the Lie algebra structure of $\Lambda^0(\Sigma) \otimes \mathfrak{g}$ in the space of parameters.

The equations of motion (5.21), (5.22) are the same as (5.29). We would like to understand this fact at the level of the action. A direct computation shows that the following equality holds:

$$S_{PL}^*(\bar{X}, A) = S_{\text{BF}} - \frac{1}{2} (dX + [A, X]) \text{Ad}_{g_X} r \text{Ad}_{g_X}^{-1} (dX + [A, X]) \quad (5.32)$$

Hence, both actions differ by terms quadratic in the equations of motion. This means that the Poisson-Lie sigma model with target G_r is equivalent, for any triangular r -matrix, to the Poisson-Lie sigma model over $G_{r=0}$, i.e. BF-theory. This is the triangular version of the connection encountered in the factorizable case ([12]), where every Poisson-Lie sigma model with target G_r for any factorizable r -matrix is (locally) equivalent to the G/G WZW model with target G .

6. D-branes in Poisson-Lie sigma models

In the previous section we have seen that the moduli spaces of solutions of the Poisson-Lie sigma models over G and $G_r \cong G^*$ coincide when g is free at the boundary (i.e. when the brane is the whole target group). We would like to find out whether such duality holds for more general boundary conditions. That is, we address the problem of finding pairs of branes $N \subset G$ and $N^* \subset G^*$ such that $\mathcal{P}(G, N) \cong \mathcal{P}(G^*, N^*)$.

Let us restrict $g|_{\partial\Sigma}$ to a closed submanifold (brane) $N \subset M$. It is natural to ask the brane N to respect the Poisson-Lie structure of G given by the r -matrix. To this end we consider a simple subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that it is r -invariant, i.e. $r\mathfrak{h} \subseteq \mathfrak{h}$. The restriction of r to \mathfrak{h} , $r|_{\mathfrak{h}}$ is an r -matrix in \mathfrak{h} . Since \mathfrak{h} is simple, its Killing form coincides (up to a constant factor) with the restriction to \mathfrak{h} of the Killing form in \mathfrak{g} . Let $H \subset G$ be the subgroup with Lie algebra \mathfrak{h} . Then, for $g \in H$, $X, Y \in \mathfrak{h}$,

$$P_g^{r|_{\mathfrak{h}}}(X, Y) = \frac{1}{2} \text{tr}(Xr|_{\mathfrak{h}}Y - X\text{Ad}_g r|_{\mathfrak{h}}\text{Ad}_g^{-1}Y) \quad (6.1)$$

defines a Poisson-Lie structure on H .

The nice point is that we can realize H^* , the dual Poisson-Lie group of H , as a subgroup of G_r . H^* is simply identified with the subgroup $H_r \subset G_r$ corresponding to the Lie subalgebra $(\mathfrak{h}, [\cdot, \cdot]_r)$ of $(\mathfrak{g}, [\cdot, \cdot]_r)$. We claim that the Poisson-Lie sigma model with target G and brane H is dual to the Poisson-Lie sigma model with target G_r and brane H_r . That is to say, there is a bulk-boundary duality between $\mathcal{P}(G, H)$ and $\mathcal{P}(G_r, H_r)$.

Before describing the duality we shall make some general considerations about the properties of these branes. The results on the boundary conditions of the fields mentioned at the end of Section 4 are written in terms of $\psi \in \Gamma(T^*\Sigma \otimes X^*T^*M)$. Let us rewrite them in terms of the field A_t appearing in the action (5.1). Here the subscript t refers to contraction of A with vectors tangent to $\partial\Sigma$.

When varying (5.1) with respect to g , a boundary term $-\int_{\Sigma} \text{dtr}(dgg^{-1} \wedge A)$ appears. Its cancellation requires $A_t \in \mathfrak{h}^{\perp}$ (\perp means orthogonal with respect to $\text{tr}(\cdot)$). On the other hand, the continuity of (5.3) at the boundary imposes $P^{\sharp}A_t \in \mathfrak{h}$. Consequently, the boundary condition for A_t is

$$A_t(\sigma) \in \mathfrak{h}^{\perp} \cap P_{g(\sigma)}^{\sharp-1}\mathfrak{h}, \quad \sigma \in \partial\Sigma \quad (6.2)$$

and the gauge transformation parameter β at the boundary is restricted by the same condition.

Condition (2.1) for Poisson-Dirac branes, applied to our present situation reads

$$\mathfrak{h} \cap P_g^{\sharp}\mathfrak{h}^{\perp} = 0, \quad \forall g \in H.$$

In particular, if H is Poisson-Dirac the gauge transformations do not act on g at the boundary.

We have that H is coisotropic if

$$P_g^{\sharp}\mathfrak{h}^{\perp} \subseteq \mathfrak{h}, \quad \forall g \in H$$

We show now that an r -invariant, simple, subgroup H is a Poisson-Dirac submanifold of G , and its dual H_r is also Poisson-Dirac in G_r . Denote by $\mathfrak{h}^\perp \subset \mathfrak{g}$ the subspace orthogonal to \mathfrak{h} with respect to $\text{tr}(\cdot)$. Firstly, the r -invariance of \mathfrak{h} implies that $P_g^\# \mathfrak{h} \subseteq \mathfrak{h}, \forall g \in H$. Using that $P^\#$ is antisymmetric we obtain that $P_g^\# \mathfrak{h}^\perp \subseteq \mathfrak{h}^\perp, \forall g \in H$. Finally, recalling that \mathfrak{h} simple $\Rightarrow \mathfrak{h} \cap \mathfrak{h}^\perp = 0$, one immediately deduces that H is Poisson-Dirac.

Observing that \mathfrak{h}_r (i.e. \mathfrak{h} equipped with the Lie bracket $[\cdot, \cdot]_{r|_{\mathfrak{h}}}$) is the same as \mathfrak{h} as a vector subspace of \mathfrak{g} and reasoning as above one shows that H_r is a Poisson-Dirac submanifold in G_r .

Finally H and H_r inherit a (smooth) Poisson structure (the Dirac bracket) from G and G_r respectively. They coincide with the Poisson structures defined by $r|_{\mathfrak{h}}$ on H and H_r (formula (6.1) for H , and analogously for H_r), making them into a pair of dual Poisson-Lie groups. Notice, however, that this induced Poisson structure does not make H (resp. H_r) into a Poisson submanifold of G (resp. G_r), so that in general it is not a Poisson-Lie subgroup, it is so if and only if it is coisotropic.

We now address the issue of the duality of the models with a pair of branes H and H_r as above. The general picture is as follows. In the case of the model over G with brane H the space of solutions, once reduced by the gauge transformations in the bulk, can be identified with the universal covering of $G_r H_d \cap H_d G_r \subset D_0$. The symplectic form Ω (see (2.9)) in D_0 (corresponding to free boundary conditions) becomes degenerate in $G_r H_d \cap H_d G_r$. This reflects the existence of gauge transformations at the boundary. Since \mathfrak{h} is r -invariant, there is a natural choice of gauge fixing for these transformations: $H_r H_d \cap H_d H_r$. The pullback of Ω to $H_r H_d \cap H_d H_r$ is nondegenerate and the infinitesimal gauge transformations span a complementary subspace to $T_p(H_r H_d \cap H_d H_r)$ in $T_p(G_r H_d \cap H_d G_r)$ for every $p \in H_r H_d \cap H_d H_r$.

The dual model over G_r with brane H_r behaves in an analogous way. The space of solutions of the equations of motion can be identified with $G_d H_r \cap H_r G_d$, but there are still gauge transformations acting on this space. The gauge fixing is given again by considering the restriction to $H_r H_d \cap H_d H_r$, which makes this model equivalent by bulk-boundary duality to the previous one.

The considerations of the previous paragraphs did not care about the existence of singular points or whether the gauge fixing is local or global. These subtleties may depend on the concrete model. Let us work out an example in which all properties of regularity and global gauge fixing are met.

Take $G = SL(n, \mathbb{C})$ with the Poisson-Lie structure (2.11) given by the standard r -matrix (3.6) and

$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in SL(n, \mathbb{C}), \quad \text{s.t.} \quad A \in SL(k, \mathbb{C}) \right\} \quad (6.3)$$

for a given $k < n$. The dual group $H_r \subset G_r \subset G \times G$ is easily described:

$$H_r = \left\{ (g_+, g_-) \in G_r, \quad \text{s.t.} \quad g_\pm = \begin{pmatrix} A_\pm & 0 \\ 0 & I \end{pmatrix}, \quad A_\pm \in SL(k, \mathbb{C}) \right\} \quad (6.4)$$

In this case,

$$\mathfrak{h}^\perp = \left\{ \begin{pmatrix} \lambda I & B \\ C & X \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{C}) \right\} \quad (6.5)$$

An easy calculation shows that $P_g^{r\sharp} \mathfrak{h}^\perp = 0$, i.e. $H \subset G$ is Poisson-Dirac and coisotropic. In particular, the inclusion map $i : H \rightarrow G$ is a Poisson map and H is a Poisson-Lie subgroup of G .

The situation is different for the dual model with target G_r and brane H_r . Recall that \mathfrak{h}_r is the same as \mathfrak{h} as a vector space, and hence also their orthogonal complements. In this case $\mathfrak{h}^\perp \cap P_{(g_+, g_-)}^{r\sharp-1} \mathfrak{h} \neq \mathfrak{h}^\perp$ and $H_r \subset G_r$ is not coisotropic. In fact, in generic points

$$\mathfrak{h}^\perp \cap P_{(g_+, g_-)}^{r\sharp-1} \mathfrak{h} = \left\{ \begin{pmatrix} \lambda I & 0 \\ 0 & X \end{pmatrix} \in \mathfrak{sl}(n, \mathbb{C}) \right\} \quad (6.6)$$

so that the brane H_r is classically admissible in the sense of Section 4.

The solutions of the equations of motion for g in the model with target G are given by

$$g(\sigma) = {}^{(h_+(\sigma), h_-(\sigma))} \hat{g} \quad (6.7)$$

where the action on \hat{g} is given by dressing transformations and $g(t, 0), g(t, \pi) \in H$. One can always take $(h_+(t, 0), h_-(t, 0)) = (e, e)$ and $(h_+(t, \pi), h_-(t, \pi)) \in H_r$. One can fix the gauge freedom for A_t at the boundary imposing $A_t = 0$. Then, h_\pm are constant at every connected component of the boundary of Σ . Therefore, the reduced phase space $\mathcal{P}(G, H)$ covers the set of pairs $((h_+, h_-), \hat{g})$ where $\hat{g} \in H$ and $(h_+, h_-) \in H_r$ and $(h_+ \hat{g}, h_- \hat{g}) \in H_r H_d \cap H_d H_r$.

For the Poisson-Lie sigma model with target G_r the solutions of the equations of motion for (g_+, g_-) are

$$(g_+(\sigma), g_-(\sigma)) = {}^{h(\sigma)} (\hat{g}_+, \hat{g}_-) \quad (6.8)$$

with $(g_+(t, 0), g_-(t, 0)), (g_+(t, \pi), g_-(t, \pi)) \in H_r$.

An analogous argument shows that $\mathcal{P}(G_r, H_r)$ is (a covering of) the set of pairs $(h, (\hat{g}_+, \hat{g}_-))$ where $(\hat{g}_+, \hat{g}_-) \in H_r$, $h \in H$ and $(h \hat{g}_+, h \hat{g}_-) \in H_d H_r \cap H_r H_d$.

The duality then exchanges degrees of freedom at the boundary with degrees of freedom in the bulk, exactly in the same way as it does for the free boundary conditions.

Notice that the duality described so far exists only for very special branes given by r -invariant subalgebras. If one considers more general situations the result is not that clean and one has, in the dual model, non-local boundary conditions that relate the fields at both connected components of $\partial\Sigma$. A more comprehensive treatment of this case will be done elsewhere.

7. Conclusions

We have extended the study of Poisson-Lie sigma models to the case of triangular r -matrices. To that end, we have presented a unified treatment of the factorizable and triangular case, that allows a convenient description of the double and the Poisson structure for the dual model in the triangular case.

We have shown that in the triangular case the dual model is equivalent, on-shell, to the BF-theory. This is reminiscent of what we have in the case of factorizable r -matrices (see [12]) where the Poisson-Lie sigma model is equivalent to the G/G Wess-Zumino-Witten model.

An aspect that is not covered in the present paper is the relation of these models with the WZ-Poisson sigma model ([14]). There are some results in the literature for the dual of a factorizable Poisson-Lie group in connection with the G/G WZW model ([21],[17]). The triangular case and the twisted version of the direct models have not been treated. We plan to turn to them in the future.

In a previous paper [6] the relation between the Poisson-Lie sigma models for pairs of dual groups was analysed uncovering a duality between the gauge degrees of freedom at the boundary in one model and those of the target field in the bulk in the other. The result in [6] was derived using free boundary conditions for both models. In a later paper [3] the question of extending the duality to more general coisotropic branes was addressed without conclusive results. In this paper we have given an answer to this question by characterizing a family of branes that can be dualized, i.e. the reduced phase space of one of the models can be mapped by a one-to-one bulk-boundary transformation into that of the dual model with a dual brane. The branes for which we have a dual brane are the subgroups whose Lie algebra is r -invariant.

Our branes do not need to be coisotropic and moreover, if we start with a coisotropic r -invariant subgroup the dual brane is an r -invariant subgroup not necessarily coisotropic, as we have shown explicitly in an example. The fact that by duality one can transform coisotropic branes into non-coisotropic ones is a strong motivation for the consideration and study of non-coisotropic submanifolds as boundary conditions for the Poisson-Sigma model.

Finally, for more general branes (i.e. beyond r -invariant subgroups) the situation is more complicated. It seems that for an arbitrary brane one obtains by duality non-local boundary conditions that relate the fields at both components of the boundary of the strip. Whether this can be interpreted as a kind of twisted periodic boundary conditions in the closed geometry will be the subject of further research.

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